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Young operators in standard orthogonal form

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Abstract. Standard (bra–ket) Young operators for $[n-1, 1]$ of S_n are expressed as tableau-permutation-separated bra and ket operators (ordered symmetrizer–antisymmetrizer products with prescribed coefficient, bra and ket oppositely ordered for the same tableau, mutually convertible by the tilde (~) transformation reversing permutation group multiplication order). Non-diagonal operators are re-expressed as tableau-permutation-separated diagonal operators. $[2\ 1^{n-2}]$ operators are obtained by star (*) transformation, multiplying each permutation by its parity, interchanging associated symmetrizers and antisymmetrizers. Non-standard (ket–bra) operators are defined in a consistent manner and identified as linear combinations of standard operators.

1. Introduction

The Young operator expressions given in this paper on the one hand form part of a general programme leading to a Young operator derivation of the base-vector expansions given in the thesis of the first author (El-Sharkaway 1975) and on the other hand aim at a simplification of Young operator expansions previously published by the second author (Jahn 1960). These latter give the Young operators of S_n as linear combinations of two-sided products of Young operators of S_{n-1} with the particular transposition $P_{n,n-1}$ and would require a long chain calculation to reach a fully explicit expression (except in the case where the operators of S_{n-1} reduce to symmetrizers or antisymmetrizers characteristic of one-dimensional representations). This previous work was general and valid for any representation of S_n. In the present paper explicit expressions in terms of symmetrizers and antisymmetrizers are given for the two particular representations $[n-1, 1]$ and $[2\ 1^{n-2}]$ of S_n. It is the intention to extend this at least to the case of the so called 'single-hook' representations $[n-k, 1^k]$ and the 'double row' representations $[n-m, m]$ and thereby arrive at an independent derivation of the expansions given by the first author (El-Sharkaway 1975).

The standard text book on Young operators is still Rutherford's (Rutherford 1948) and an important relevant paper is that of McIntosh (McIntosh 1960).

Young operators have application not only to many-body problems but also to tensor symmetrization problems connected with the continuous group representations.

2. Young operators for the representation $[n-1, 1]$ and its dual $[2\ 1^{n-2}]$

The $(n-1)^2$ Young operators o^n_{ab} $(a, b = 2, 3, \ldots, n)$ for the representation $[n-1, 1]$ of

S_n in standard orthogonal form are required to satisfy

$$o^n_{ab}o^n_{cd}=\delta_{bc}o^n_{ad}, \tag{2.1}$$

$$P_{a,a+1}o^n_{ab}=(1/a)o^n_{ab}+\{(a^2-1)^{1/2}/a\}o^n_{a+1,b}, \tag{2.2}$$

$$o^n_{ab}P_{b,b+1}=(1/b)o^n_{ab}+\{(b^2-1)^{1/2}/b\}o^n_{a,b+1}. \tag{2.3}$$

Here the numerical label a is an abbreviation for the standard Young tableau label

$$a_n=\frac{1\ldots\dot{a}\ldots n}{a}=\frac{12\ldots a-1\;a+1\ldots n-1n}{a}, \tag{2.4}$$

where, following the first author's notation (El-Sharkaway 1975), $\dot{a}$ is used to denote the omission of a from $2\ldots n$. The coefficient $(1/a)$ occurs in (2.2) because the Young axial distance from $a+1$ to a in the Young tableau (2.4) is $+a$.

The $(n-1)^2$ Young operators $o^n_{a^*b^*}$ $(a, b=2, 3, \ldots, n)$ for the representation $[2\,1^{n-2}]$ dual to $[n-1, 1]$ are required to satisfy

$$o^n_{a^*b^*}o^n_{c^*d^*}=\delta_{bc}o^n_{a^*d^*} \tag{2.1*}$$

$$P_{a,a+1}o^n_{a^*b^*}=-(1/a)o^n_{a^*b^*}+\{(a^2-1)^{1/2}/a\}o^n_{(a+1)^*b^*} \tag{2.2*}$$

$$o^n_{a^*b^*}P_{b,b+1}=-(1/b)o^n_{a^*b^*}+\{(b^2-1)^{1/2}/b\}o^n_{a^*(b+1)^*}. \tag{2.3*}$$

Here the starred numerical label a^* is an abbreviation for the standard Young tableau label

$$a^*_n=1a*\ldots\dot{a}\ldots n=1a*2\ldots a-1\;a+1\ldots n-1\;n$$
$$=\begin{matrix}1 & a,\\ \vdots & \\ a-1 & \\ a+1 & \\ \vdots & \\ n & \end{matrix} \tag{2.4*}$$

where, using again the first author's notation (El-Sharkaway 1975) a star is employed to avoid the printing of a column. The coefficient $-(1/a)$ occurs in (2.2)* because the Young axial distance from $a+1$ to a in the Young tableau (2.4)* is $-a$.

3. Symmetrizers and antisymmetrizers: the star transformation

The symmetrizer

$$S_{1\ldots n}=(1/n!)\sum_{\text{all } n!P \text{ in } S_n} P$$
$$=\{(I+P_{1n}+P_{2n}+\ldots+P_{n-1,n})/n\}S_{1\ldots\dot{n}}$$
$$=S_{1\ldots\dot{n}}\{(I+P_{1n}+P_{2n}+\ldots+P_{n-1,n})/n\} \tag{3.1}$$

is the Young operator for the totally symmetric representation $[n]$ of S_n and, being totally symmetric, satisfies

$$S_{1\ldots n} = PS_{1\ldots n} = S_{1\ldots n}P \qquad (P \text{ in } S_n)$$
$$\Rightarrow S_{1\ldots a}S_{1\ldots n} = S_{1\ldots n}S_{1\ldots a} \qquad (a \leqslant n). \tag{3.2}$$

The antisymmetrizer

$$A_{1\ldots n} = (1/n!) \sum_{\text{all } n!P \text{ in } S_n} \epsilon_P P \qquad (\epsilon_P = \pm 1, \text{parity of } P)$$
$$= \{(I - P_{1n} - P_{2n} - \ldots - P_{n-1,n})/n\}A_{1\ldots \dot{n}}$$
$$= A_{1\ldots \dot{n}}\{(I - P_{1n} - P_{2n} - \ldots - P_{n-1,n})/n\} \tag{3.1)*}$$

is the Young operator for the totally antisymmetric representation $[1^n]$ dual to $[n]$ and, being totally antisymmetric, satisfies

$$A_{1\ldots n} = \epsilon_P PA_{1\ldots n} = A_{1\ldots n}\epsilon_P P \qquad (\epsilon_P = \pm 1, \text{parity of } P, P \text{ in } S_n)$$
$$= A_{1\ldots a}A_{1\ldots n} = A_{1\ldots n}A_{1\ldots a} \qquad (a \leqslant n). \tag{3.2)*}$$

(3.1)*, (3.2)* are obtained from (3.1), (3.2) by what we call the star (*) transformation which multiplies each permutation P by its parity ϵ_P and hence interchanges S and A throughout.

If, with a slight generalization of notation, we write

$$S_{1a} = (I + P_{1a})/2, \qquad A_{1a} = (I - P_{1a})/2, \tag{3.3), (3.3)*}$$

so that we have

$$A_{1\ldots a} = A_{1\ldots \dot{a}}A_{1a} = A_{1a}A_{1\ldots \dot{a}}, \tag{3.4}$$
$$S_{1\ldots n} = S_{1a}S_{1\ldots \dot{a}\ldots n} = S_{1\ldots \dot{a}\ldots n}S_{1a}, \tag{3.5}$$

then the pair of relations

$$A_{1\ldots a}S_{1\ldots n} = S_{1\ldots n}A_{1\ldots a} = 0 = S_{1\ldots a}A_{1\ldots n} = A_{1\ldots n}S_{1\ldots a} \tag{3.6), (3.6)*}$$

are a direct consequence of the basic pair

$$A_{1a}S_{1a} = (I - P_{1a})(I + P_{1a})/4 = 0 = (I + P_{1a})(I - P_{1a})/4 = S_{1a}A_{1a}. \tag{3.7), (3.7)*}$$

4. Reduction of multiple symmetrizer–antisymmetrizer products: star, tilde and star–tilde transformations

We show that the following set of four relations holds:

$$A_{1a}S_{1\ldots \dot{a}\ldots b} = 2K_{/bb}(A_{1a}S_{1\ldots \dot{a}\ldots b})(A_{1a}S_{1\ldots \dot{a}\ldots b}), \tag{4.1}$$
$$S_{1a}A_{1\ldots \dot{a}\ldots b} = 2K_{/bb}(S_{1a}A_{1\ldots \dot{a}\ldots b})(S_{1a}A_{1\ldots \dot{a}\ldots b}), \tag{4.1)*}$$
$$S_{1\ldots \dot{a}\ldots b}A_{1a} = 2K_{/bb}(S_{1\ldots \dot{a}\ldots b}A_{1a})(S_{1\ldots \dot{a}\ldots b}A_{1a}), \tag{\widetilde{4.1}}$$
$$A_{1\ldots \dot{a}\ldots b}S_{1a} = 2K_{/bb}(A_{1\ldots \dot{a}\ldots b}S_{1a})(A_{1\ldots \dot{a}\ldots b}S_{1a}). \tag{\widetilde{4.1})*}$$

Here, in accordance with the notation

$$K_{/an} = \{(a-1)(n-1)/(an)\}^{1/2} \tag{4.2}$$

for a numerical coefficient constantly occurring later, we have

$$K_{/bb} = (b-1)/b. \tag{4.3}$$

We have already used a star (*) to designate the transformation which interchanges A and S throughout, equivalent to multiplying each permutation P by its parity ϵ_P. We now use a tilde (~) to designate the transformation which reverses the order of the S and A factors throughout on both sides of the equation, equivalent to a reversal in sense of permutation group multiplication. Since any algebraic identity involving sums and products of permutations will remain an identity when each permutation is multiplied by its parity and likewise remain an identity when the sense of group multiplication is reversed, we see that equations (4.1)*, $(\widetilde{4.1})$ and $(\widetilde{4.1})$* obtained from (4.1) by means respectively of star, tilde and combined star–tilde transformations will have been proved when (4.1) is established.

(4.1) may be rewritten as

$$[A_{1a}S_{1\ldots\dot{a}\ldots b} - bI/\{2(b-1)\}](A_{1a}S_{1\ldots\dot{a}\ldots b}) = 0. \tag{4.4}$$

Writing, from (3.1),

$$S_{1\ldots\dot{a}\ldots b} = \{(I+P_{12}+\ldots+P_{1\dot{a}}+\ldots+P_{1b})/(b-1)\}S_{2\ldots\dot{a}\ldots b}, \tag{4.5}$$

commuting $S_{2\ldots\dot{a}\ldots b}$ with A_{1a}, using (from (3.2))

$$S_{2\ldots\dot{a}\ldots b}S_{1\ldots\dot{a}\ldots b} = S_{1\ldots\dot{a}\ldots b}, \tag{4.6}$$

writing, from (3.3)*, $A_{1a} = (I-P_{1a})/2$ and removing the common factor $1/\{2(b-1)\}$ there remains

$$[(I-P_{1a})(I+P_{12}+\ldots+P_{1\dot{a}}+\ldots P_{1b}) - bI]A_{1a}S_{1\ldots\dot{a}\ldots b} = 0. \tag{4.7}$$

Now

$$P_{1a}P_{1j} = P_{a1j} = P_{ja1} = P_{aj}P_{1a} \qquad (j = 2, \ldots, \dot{a}, \ldots, b) \tag{4.8}$$

and

$$-P_{1a}A_{1a} = +A_{1a}. \tag{4.9}$$

There remains

$$[I+P_{12}+\ldots+P_{1\dot{a}}+\ldots+P_{1b}+I+P_{a2}+\ldots+P_{a\dot{a}}+\ldots+P_{ab} - bI]A_{1a}S_{1\ldots\dot{a}\ldots b} = 0, \tag{4.10}$$

or, changing the overall sign,

$$\sum_{j=2\ldots\dot{a}\ldots b} (I-P_{1j}-P_{aj})A_{1a}S_{1\ldots\dot{a}\ldots b} = 0. \tag{4.11}$$

Now (3.1)* is consistent with

$$3A_{1aj} = (I-P_{1j}-P_{aj})A_{1a} \qquad (j = 2, \ldots, \dot{a}, \ldots, b) \tag{4.12}$$

and hence (4.1) follows from

$$A_{1aj}S_{1\ldots\dot{a}\ldots b} = A_{1aj}\underline{A_{1j}S_{1j}}S_{1\ldots\dot{a}\ldots b} = 0, \qquad (j = 2, \ldots, \dot{a}, \ldots, b) \tag{4.13}$$

by (3.7).

5. Bra and ket tableau operators: tilde transforms of each other

Bra and ket tableau operators are so defined that they are mutual tilde transforms of each other. Thus, for $[n-1, 1]$, with $K_{/an}$ as in (4.2) and with $a = 2, 3, \dots, n$, the bra and ket tableau operators are defined to be respectively

$$\langle a_n| = \left\langle \begin{matrix} 1 & \dots & \dot{a} & \dots & n \\ a & & & & \end{matrix} \right| = |\tilde{a}_n\rangle = 2K_{/an} S_{1\dots\dot{a}} A_{1a} S_{1\dots\dot{a}\dots n}, \tag{5.1}$$

$$|a_n\rangle = \left| \begin{matrix} 1 & \dots & \dot{a} & \dots & n \\ a & & & & \end{matrix} \right\rangle = \langle \tilde{a}_n| = 2K_{/an} S_{1\dots\dot{a}\dots n} A_{1a} S_{1\dots\dot{a}}. \tag{$\widetilde{5.1}$}$$

The corresponding operators for $[2\ 1^{n-2}]$ are obtained from the above by star transformation:

$$\langle a_n^*| = \langle 1a * \dots \dot{a} \dots n| = |\tilde{a}_n^*\rangle = 2K_{/an} A_{1\dots\dot{a}} S_{1a} A_{1\dots\dot{a}\dots n}, \tag{5.1)*}$$

$$|a_n^*\rangle = |1a * \dots \dot{a} \dots n\rangle = \langle \tilde{a}_n^*| = 2K_{/an} A_{1\dots\dot{a}\dots n} S_{1a} A_{1\dots\dot{a}}. \tag{$\widetilde{5.1}$)*}$$

For the particular case $a = n$ the expressions become symmetric with respect to tilde transformation so that, with

$$K_{/nn} = (n-1)/n \tag{5.2}$$

$$\langle n_n| = \left\langle \begin{matrix} 1 & \dots & \dot{n} \\ n & & \end{matrix} \right| = \left| \begin{matrix} 1 & \dots & \dot{n} \\ n & & \end{matrix} \right\rangle = |n_n\rangle = \langle \tilde{n}_n| = |\tilde{n}_n\rangle = 2K_{/nn} S_{1\dots\dot{n}} A_{1n} S_{1\dots\dot{n}}. \tag{5.3), ($\widetilde{5.3}$}$$

$$\langle n_n^*| = \langle 1n * \dots \dot{n}| = |1n * \dots \dot{n}\rangle = |n_n^*\rangle = \langle \tilde{n}_n^*| = |\tilde{n}_n^*\rangle = 2K_{/nn} A_{1\dots\dot{n}} S_{1n} A_{1\dots\dot{n}}. \tag{5.3)*, ($\widetilde{5.3}$)*}$$

We may put $n = a$ in these expressions and obtain, with

$$K_{/aa} = (a-1)/a, \tag{5.4}$$

the following special tableau operators for $[a-1, 1]$ and $[2\ 1^{a-2}]$:

$$\langle a_a| = \left\langle \begin{matrix} 1 & \dots & \dot{a} \\ a & & \end{matrix} \right| = \left| \begin{matrix} 1 & \dots & \dot{a} \\ a & & \end{matrix} \right\rangle = |a_a\rangle = \langle \tilde{a}_a| = |\tilde{a}_a\rangle = 2K_{/aa} S_{1\dots\dot{a}} A_{1a} S_{1\dots\dot{a}}, \tag{5.5), ($\widetilde{5.5}$}$$

$$\langle a_a^*| = \langle 1a * \dots \dot{a}| = |1a * \dots \dot{a}\rangle = |a_a^*\rangle = \langle \tilde{a}_a^*| = |\tilde{a}_a^*\rangle = 2K_{/aa} A_{1\dots\dot{a}} S_{1a} A_{1\dots\dot{a}}. \tag{5.5)*, ($\widetilde{5.5}$)*}$$

These special cases are considered again in the next section.

6. Diagonal bra–ket and ket–bra operators

Diagonal bra–ket and ket–bra operators are defined as simple products of the corresponding bra and ket operators. Included in our main theorem is the statement that standard diagonal Young operators are equal to corresponding diagonal bra–ket operators, whilst the non-standard diagonal ket–bra operators, whose expression simplifies, are in general linear combinations of both diagonal and non-diagonal standard Young operators.

Thus with $K_{/nn}$, $K_{/aa}$ as defined in (5.2), (5.4) we find from the equation set (5.1) for $[n-1, 1]$ and $[2\ 1^{n-2}]$ respectively

$$o^n_{aa} = \langle a_n | a_n \rangle = \left\langle \begin{matrix} 1 \dots \dot{a} \dots n \\ a \end{matrix} \middle| \begin{matrix} 1 \dots \dot{a} \dots n \\ a \end{matrix} \right\rangle$$

$$= 4K_{/aa}K_{/nn}S_{1\dots\dot{a}}A_{1a}S_{1\dots\dot{a}\dots n}A_{1a}S_{1\dots\dot{a}}, \tag{6.1}$$

$$o^n_{a^*a^*} = \langle a^*_n | a^*_n \rangle = \langle 1a * \dots \dot{a} \dots n | 1a * \dots \dot{a} \dots n \rangle$$

$$= 4K_{/aa}K_{/nn}A_{1\dots\dot{a}}S_{1a}A_{1\dots\dot{a}\dots n}S_{1a}A_{1\dots\dot{a}}. \tag{6.1)*$$

Although the non-standard diagonal ket–bra operators are obtained from (6.1), (6.1)* by a tilde transformation, it is to be noted that they are *not* obtained by direct tilde transformation of the final expressions (which are in fact invariant with respect to overall symmetrizer–antisymmetrizer order reversal): the transformation is carried out indirectly by action on the bra and ket parts separately. To emphasize that this operation in a sense cuts the o operator in half and reverses the two halves we use an x for the resulting non-standard diagonal ket–bra operator, writing, for $[n-1, 1]$,

$$x^n_{aa} = |a_n\rangle\langle a_n| = \left| \begin{matrix} 1 \dots \dot{a} \dots n \\ a \end{matrix} \right\rangle \left\langle \begin{matrix} 1 \dots \dot{a} \dots n \\ a \end{matrix} \right| = \langle \tilde{a}_n | \tilde{a}_n \rangle$$

$$= 4K_{/aa}K_{/nn}S_{1\dots\dot{a}\dots n}\underline{A_{1a}S_{1\dots\dot{a}}A_{1a}S_{1\dots\dot{a}}}S_{1\dots\dot{a}\dots n}$$

$$= 2K_{/nn}S_{1\dots\dot{a}\dots n}A_{1a}S_{1\dots\dot{a}\dots n} \tag{6.1)'$$

and by star transformation, for $[2\ 1^{n-2}]$,

$$x^n_{a^*a^*} = |a^*_n\rangle\langle a^*_n| = |1a * \dots \dot{a} \dots n\rangle\langle 1a * \dots \dot{a} \dots n| = \langle \tilde{a}^*_n | \tilde{a}^*_n \rangle$$

$$= 2K_{/nn}A_{1\dots\dot{a}\dots n}S_{1a}A_{1\dots\dot{a}\dots n}. \tag{6.1)'*$$

A consistent extension of these definitions to the non-diagonal case is given later.

We have already seen, in (5.3), $(\widetilde{5.3})$, that the special limiting bra and ket of $[n-1, 1]$ having $a = n$ are identical; it follows that they commute and hence the bra–ket and ket–bra diagonal operators are equal also. We find in fact that the expression for the product reduces in such a way to make bra, ket, bra–ket and ket–bra all equal in this limiting case. Thus, for $[n-1, 1]$,

$$o^n_{nn} = \left\langle \begin{matrix} 1 \dots \dot{n} \\ n \end{matrix} \middle| \begin{matrix} 1 \dots \dot{n} \\ n \end{matrix} \right\rangle = \langle n_n | n_n \rangle = |n_n\rangle\langle n_n| = \left| \begin{matrix} 1 \dots \dot{n} \\ n \end{matrix} \right\rangle \left\langle \begin{matrix} 1 \dots \dot{n} \\ n \end{matrix} \right| = x^n_{nn}$$

$$= 4(K_{/nn})^2\underline{S_{1\dots\dot{n}}A_{1n}S_{1\dots\dot{n}}A_{1n}}S_{1\dots\dot{n}}$$

$$= 2K_{/nn}S_{1\dots\dot{n}}A_{1n}S_{1\dots\dot{n}}$$

$$= \left\langle \begin{matrix} 1 \dots \dot{n} \\ n \end{matrix} \right| = \langle n_n | = | n_n \rangle = \left| \begin{matrix} 1 \dots \dot{n} \\ n \end{matrix} \right\rangle \tag{6.2), $(\widetilde{6.2})$$

and, by star transformation, for $[2\ 1^{n-2}]$,

$$o^n_{n^*n^*} = \langle 1n * \dots \dot{n} | 1n * \dots \dot{n} \rangle = \langle n^*_n | n^*_n \rangle = |n^*_n\rangle\langle n^*_n|$$

$$= |1n * \dots \dot{n}\rangle\langle 1n * \dots \dot{n}| = x^n_{n^*n^*}$$

$$= 2K_{/nn}A_{1\dots\dot{n}}S_{1n}A_{1\dots\dot{n}}$$

$$= \langle 1n * \dots \dot{n} | = \langle n^*_n | = | n^*_n \rangle = | 1n * \dots \dot{n} \rangle. \qquad (6.2)^*,\ (\widetilde{6.2})^*$$

We establish now the following relations for $[n-1, 1]$

$$\langle a_n| = \langle a_n|a_n\rangle\langle a_n| = o^n_{aa}\langle a_n| = \langle a_n|x^n_{aa}, \tag{6.3}$$

$$|a_n\rangle = |a_n\rangle\langle a_n|a_n\rangle = |a_n\rangle o^n_{aa} = x^n_{aa}|a_n\rangle, \tag{$\widetilde{6.3}$}$$

from which follow

$$o^n_{aa} = \langle a_n|a_n\rangle = \langle a_n|a_n\rangle\langle a_n|a_n\rangle = o^n_{aa}o^n_{aa} = \langle a_n|x^n_{aa}|a_n\rangle \tag{6.4}$$

$$x^n_{aa} = |a_n\rangle\langle a_n| = |a_n\rangle\langle a_n|a_n\rangle\langle a_n| = x^n_{aa}x^n_{aa} = |a_n\rangle o^n_{aa}\langle a_n| \tag{$\widetilde{6.4}$}$$

showing that o^n_{aa} and x^n_{aa} are idempotent. Similar relations are obtained for $[2\ 1^{n-2}]$ by star transformation. Using the simpler expression (6.1)′ for x^n_{aa} we have

$$\langle a_n|x^n_{aa} = 4K_{/an}K_{/nn}S_{1\ldots\dot{a}}\underline{A_{1a}S_{1\ldots\dot{a}\ldots n}A_{1a}S_{1\ldots\dot{a}\ldots n}} = 2K_{/an}S_{1\ldots\dot{a}}A_{1a}S_{1\ldots\dot{a}\ldots n} = \langle a_n|, \tag{6.3$'$}$$

which in effect proves the whole of (6.3) and the rest follow. In short:

$$\left.\begin{matrix}\text{bra}\langle a_n| \\ \text{ket}|a_n\rangle\end{matrix}\right\} \text{ is a right (left) -hand eigenstate of } \left\{\begin{matrix} o^n_{aa}(x^n_{aa}) \\ x^n_{aa}(o^n_{aa})\end{matrix}\right. \tag{6.3$''$}$$

with eigenvalue +1 in all cases.

Putting $n = a$ in (6.1), (6.1)′ and using (5.5) we find

$$o^a_{aa} = \langle a_a|a_a\rangle = |a_a\rangle\langle a_a| = x^a_{aa} = \langle a_a| = |a_a\rangle = 2K_{/aa}S_{1\ldots\dot{a}}A_{1a}S_{1\ldots\dot{a}}. \tag{6.5}$$

Then from (5.1), ($\widetilde{5.1}$) we find

$$\begin{aligned}|a_a\rangle\langle a_n| = o^a_{aa}\langle a_n| &= 4K_{/aa}K_{/an}\underline{S_{1\ldots\dot{a}}A_{1a}S_{1\ldots\dot{a}}A_{1a}}S_{1\ldots\dot{a}\ldots n} \\ &= 2K_{/an}S_{1\ldots\dot{a}}A_{1a}S_{1\ldots\dot{a}\ldots n} = \langle a_n|,\end{aligned} \tag{6.6}$$

$$\begin{aligned}|a_n\rangle\langle a_a| = |a_n\rangle o^a_{aa} &= 4K_{/aa}K_{/an}S_{1\ldots\dot{a}\ldots n}\underline{A_{1a}S_{1\ldots\dot{a}}A_{1a}S_{1\ldots\dot{a}}} \\ &= 2K_{/an}S_{1\ldots\dot{a}\ldots n}A_{1a}S_{1\ldots\dot{a}} = |a_n\rangle,\end{aligned} \tag{$\widetilde{6.6}$}$$

noting that the tilde transformation, apart from replacing bra by ket involves also a reversal in the operator product order. These relations may be summarized as

$$\left.\begin{matrix}\text{bra}\langle a_n| \text{ is right-hand} \\ \text{ket}|a_n\rangle \text{ is left-hand}\end{matrix}\right\} \text{eigenstate of } o^a_{aa}. \tag{6.7}$$

Tilde transformation without change of operator product order leads to another result:

$$\langle a_a|a_n\rangle = 4K_{/aa}K_{/an}S_{1\ldots\dot{a}}A_{1a}S_{1\ldots\dot{a}\ldots n}A_{1a}S_{1\ldots\dot{a}} = \langle a_n|a_a\rangle = K_{n/a}\langle a_n|a_n\rangle \tag{6.8}$$

or,

$$\langle a_n|a_n\rangle = K_{a/n}\langle a_a|a_n\rangle = K_{a/n}\langle a_n|a_a\rangle, \tag{6.9}$$

with

$$K_{n/a} = K_{/an}/K_{/nn} = [n(a-1)/\{(n-1)a\}]^{1/2}, \tag{6.10}$$

$$K_{a/n} = 1/K_{n/a} = [a(n-1)/\{(a-1)n\}]^{1/2}. \tag{6.11}$$

This result may be presented also as

$$o^{a}_{aa}|a_n\rangle = \langle a_n|o^{a}_{aa} = K_{n/a}o^{n}_{aa}. \tag{6.12}$$

Similar results are obtainable, of course, for $[2\ 1^{n-2}]$ by star transformation.

7. Difference formula for the limiting operators

We show that the limiting operators o^{n}_{nn} of $[n-1, 1]$ (see (6.2), $(\widetilde{6.2})$) may be expressed simply as the difference of two symmetrizers. We have

$$\begin{aligned} S_{1\ldots\dot n} - S_{1\ldots n} &= S_{1\ldots\dot n}\{I - (I + P_{1n} + P_{2n} + \ldots + P_{n-1,n})/n\}S_{1\ldots\dot n} \\ &= (1/n)S_{1\ldots\dot n}\{(n-1)I - (P_{1n} + P_{2n} + \ldots + P_{n-1,n})\}S_{1\ldots\dot n} \\ &= (2/n)S_{1\ldots\dot n}(A_{1n} + A_{2n} + \ldots + A_{n-1,n})S_{1\ldots\dot n}. \end{aligned} \tag{7.1}$$

Now, for $2 \leqslant a \leqslant n-1$, we have

$$S_{1\ldots a\ldots\dot n}A_{an}S_{1\ldots a\ldots\dot n} = S_{1\ldots a\ldots\dot n}P_{1a}A_{an}P^{-1}_{1a}S_{1\ldots a\ldots\dot n} = S_{1\ldots\dot n}A_{1n}S_{1\ldots\dot n}. \tag{7.2}$$

Hence

$$S_{1\ldots\dot n} - S_{1\ldots n} = 2\{(n-1)/n\}S_{1\ldots\dot n}A_{1n}S_{1\ldots\dot n} = 2K_{/nn}S_{1\ldots\dot n}A_{1n}S_{1\ldots\dot n} = o^{n}_{nn}. \tag{7.3}$$

By star transformation, it follows immediately, for $[2\ 1^{n-2}]$,

$$A_{1\ldots\dot n} - A_{1\ldots n} = o^{n}_{n^*n^*}. \tag{7.3*}$$

8. Orthogonality of ket–bra symbols for $[n-1, 1]$

Since, from (5.1), $(\widetilde{5.1})$,

$$\begin{aligned} |a_n\rangle &= 2K_{/an}S_{1\ldots\dot a\ldots n}A_{1a}S_{1\ldots\dot a}, \\ \langle b_n| &= 2K_{/bn}S_{1\ldots\dot b}A_{1b}S_{1\ldots\dot b\ldots n}, \end{aligned} \tag{8.1}$$

we see that $|a_n\rangle\langle b_n|$ involves the product

$$Q_{ab} = A_{1a}S_{1\ldots\dot a}S_{1\ldots\dot b}A_{1b}. \tag{8.2}$$

Now if $a < b$, we have, by (3.2),

$$S_{1\ldots\dot a}S_{1\ldots\dot b} = S_{1\ldots\dot b}, \tag{8.3}$$

so that

$$Q^{ab}_{(a<b)} = A_{1a}S_{1\ldots\dot b}A_{1b} = \underline{A_{1a}S_{1a}}S_{1\ldots\dot b}A_{1b} = 0. \tag{8.4}$$

On the other hand, if $a > b$, we have

$$S_{1\ldots\dot a}S_{1\ldots\dot b} = S_{1\ldots\dot a}, \tag{8.5}$$

so that

$$Q^{ab}_{(a>b)} = A_{1a}S_{1\ldots\dot a}A_{1b} = A_{1a}S_{1\ldots\dot a}\underline{S_{1b}A_{1b}} = 0. \tag{8.6}$$

Thus

$$|a_n\rangle\langle b_n| = \delta_{ab}|a_n\rangle\langle a_n| = \delta_{ab}x^n_{aa} = \delta_{ab}2K_{/nn}S_{1\ldots\dot{a}\ldots n}A_{1a}S_{1\ldots\dot{a}\ldots n}. \quad (8.7)$$

It follows, by star transformation, for $[2\ 1^{n-2}]$,

$$|a^*_n\rangle\langle b^*_n| = \delta_{ab}2K_{/nn}A_{1\ldots\dot{a}\ldots n}S_{1a}A_{1\ldots\dot{a}\ldots n}. \quad (8.8)$$

9. Tableau permutations for $[n-1, 1]$

We denote the permutation which converts the limiting tableau

$$n_n = \begin{matrix} 1\,2\ldots a-1\;a\;a+1\ldots n-2\;n-1 \\ n \end{matrix} \quad (9.1)$$

into the general tableau

$$a_n = \begin{matrix} 1\,2\ldots a-1\;a+1\;a+2\ldots n-1\;n \\ a \end{matrix} \quad (9.2)$$

by

$$(a_n|P|n_n) = P_{(a,a+1,a+2,\ldots,n-2,n-1,n)} \quad (9.3)$$

and the inverse permutation, converting a_n into n_n by

$$(n_n|P|a_n) = P_{(n,n-1,n-2,\ldots,a+2,a+1,a)}. \quad (9.4)$$

The permutation which converts

$$b_n = \begin{matrix} 1\ldots\dot{b}\ldots n \\ b \end{matrix} \quad \text{into} \quad a_n = \begin{matrix} 1\ldots\dot{a}\ldots n \\ a \end{matrix}$$

may then be written as

$$(a_n|P|b_n) = (a_n|P|n_n)(n_n|P|b_n) = P_{(a,a+1,\ldots,n)}P_{(n,n-1,\ldots,b)}. \quad (9.5)$$

This may be evaluated for the two cases $a < b$ and $a > b$ as follows:

If $a < b$

$$P_{(a,a+1,\ldots n)} = P_{(a,a+1,\ldots,b)}P_{(b,b+1,\ldots,n)} = P_{(a,a+1,\ldots,b)}P^{-1}_{(n,n-1,\ldots,b)}, \quad (9.6)$$

so that

$$(a_n|P|b_n)_{(a<b)} = P_{(a,a+1,\ldots,b)}. \quad (9.7)$$

If $a > b$

$$P_{(n,n-1,\ldots,b)} = P_{(n,n-1,\ldots,a)}P_{(a,a-1,\ldots,b)} = P^{-1}_{(a,a+1,\ldots,n)}P_{(a,a-1,\ldots,b)},$$

so that

$$(a_n|P|b_n)_{(a>b)} = P_{(a,a-1,\ldots,b)}. \quad (9.8)$$

It is easy to verify this directly from the form the tableaux a_n, b_n have in two cases:

$a<b$

$$b_n = \begin{matrix} 1\,2 \ldots a-1\; a\; a+1 \ldots b-1\; b+1 \ldots n \\ b \end{matrix}$$

$$a_n = \begin{matrix} 1\,2 \ldots a-1\; a+1\; a+2 \ldots b\; b+1 \ldots n \\ a \end{matrix}$$

$$(a_n|P|b_n) = P_{(a,a+1,a+2,\ldots,b-1,b)}. \qquad (9.9)$$

$a>b$

$$b_n = \begin{matrix} 1\,2 \ldots b-1\; b+1 \ldots a-1\; a\; a+1 \ldots n \\ b \end{matrix}$$

$$a_n = \begin{matrix} 1\,2 \ldots b-1\; b \ldots a-2\; a-1\; a+1 \ldots n \\ a \end{matrix}$$

$$(a_n|P|b_n) = P_{(a,a-1,a-2,\ldots,b+1,b)}. \qquad (9.10)$$

10. Postulated expression for non-diagonal Young operators

We postulate that the standard non-diagonal Young operators for the representation $[n-1, 1]$ in orthgonal form are given by the following bra operator–tableau permutation–ket operator product:

$$o^n_{ab} = \langle a_n|(a_n|P|b_n)|b_n\rangle, \qquad (a, b = 2, 3, \ldots, n). \qquad (10.1)$$

Since $(a_n|P|a_n) = I$, the identity, this includes the already postulated diagonal case:

$$o^n_{aa} = \langle a_n|a_n\rangle, \qquad (a = 2, 3, \ldots, n). \qquad (10.2)$$

By means of the following lemma we arrive at a new expression for the non-standard diagonal ket–bra operators x^n_{aa} which suggests a generalization to the non-standard non-diagonal case.

Lemma

$$P_{(a,a+1,\ldots,n)}\langle n_n|n_n\rangle = |a_n\rangle\langle a_n|P_{(a,a+1,\ldots,n)}, \qquad (10.3)$$

$$\langle n_n|n_n\rangle P_{(n,n-1,\ldots,a)} = P_{(n,n-1,\ldots,a)}|a_n\rangle\langle a_n|, \qquad (\widetilde{10.3})$$

where, in accordance with tilde transformation reversing permutation group multiplication order, the transformation which converts (10.3) into $(\widetilde{10.3})$ reverses the order of the operators, replaces a permutation P by its inverse P^{-1} and, as has already been explained, replaces each bra operator by its corresponding ket operator, each ket operator by its corresponding bra operator.

Proof. From (6.2), $(\widetilde{6.2})$ we have

$$\langle n_n|n_n\rangle = 2K_{/nn}S_{1\ldots\dot{n}}A_{1n}S_{1\ldots\dot{n}}. \qquad (6.2), (\widetilde{6.2})$$

Hence, noting particularly that transformation by a permutation, e.g. by

$$P = P_{(a,a+1,\dots,n)} = P^{-1}_{(n,n-1,\dots,a)}, \tag{10.4}$$

leaves a numerical coefficient such as $K_{/nn}$ unaltered,

$$P_{(a,a+1,\dots,n)}\langle n_n|n_n\rangle P^{-1}_{(a,a+1,\dots,n)}$$
$$= P^{-1}_{(n,n-1,\dots,a)}\langle n_n|n_n\rangle P_{(n,n-1,\dots,a)} = 2K_{/nn}S_{1\dots\dot{a}\dots n}A_{1a}S_{1\dots\dot{a}\dots n} = |a_n\rangle\langle a_n|, \tag{10.5}$$

by (6.1)′, which establishes both (10.3) and $(\widetilde{10.3})$. We see that (10.5) expresses the non-standard diagonal ket–bra operator x^n_{aa} associated with $[n-1, 1]$ as linear combinations of standard $[n-1, 1]$ Young operators obtained by simultaneous left- and right-handed application of permutations to the limiting operator o^n_{nn}:

$$x^n_{aa} = |a_n\rangle\langle a_n| = P_{(a,a+1,\dots,n)}o^n_{nn}P_{(n,n-1,\dots,a)}. \tag{10.6}$$

This suggests immediately the following generalization:

$$x^n_{ab} = P_{(a,a+1,\dots,n)}o^n_{nn}P_{(n,n-1,\dots,b)}. \tag{10.7}$$

Accepting this new postulate, we find, using (10.3), $(\widetilde{10.3})$

$$\begin{aligned} x^n_{ab} &= (P_{(a,a+1,\dots,n)}o^n_{nn})(o^n_{nn}P_{(n,n-1,\dots,b)}) \\ &= |a_n\rangle\langle a_n|P_{(a,a+1,\dots,n)}P_{(n,n-1,\dots,b)}|b_n\rangle\langle b_n| \\ &= |a_n\rangle\langle a_n|(a_n|P|b_n)|b_n\rangle\langle b_n| \\ &= |a_n\rangle o^n_{ab}\langle b_n|, \end{aligned} \tag{10.8}$$

using (10.1). Conversely, assuming o^n_{ab} has Young operator properties,

$$\begin{aligned} o^n_{ab} &= o^n_{aa}o^n_{ab}o^n_{bb} \\ &= \langle a_n|a_n\rangle\langle a_n|(a_n|P|b_n)|b_n\rangle\langle b_n|b_n\rangle \\ &= \langle a_n|x^n_{aa}P_{(a,a+1,\dots,n)}P_{(n,n-1,\dots,b)}x^n_{bb}|b_n\rangle \\ &= \langle a_n|P_{(a,a+1,\dots,n)}o^n_{nn}o^n_{nn}P_{(n,n-1,\dots,b)}|b_n\rangle \\ &= \langle a_n|x^n_{ab}|b_n\rangle, \end{aligned} \tag{10.8$'$}$$

by (10.7). It may be noted that (10.8)′ is not the tilde transform of (10.8) as there is no reversal of the operator order. In fact from (10.7) we see that

$$\tilde{x}^n_{ab} = P^{-1}_{(n,n-1,\dots,b)}\tilde{o}^n_{nn}P^{-1}_{(a,a+1,\dots,n)} = P_{(b,b+1,\dots,n)}o^n_{nn}P_{(n,n-1,\dots,a)} = x^n_{ba} = |b_n\rangle o^n_{ba}\langle a_n| \tag{$\widetilde{10.8}$}$$

and likewise

$$\tilde{o}^n_{ab} = \langle b_n|(a_n|P|b_n)^{-1}|a_n\rangle = \langle b_n|(b_n|P|a_n)|a_n\rangle = o^n_{ba} = \langle b_n|x^n_{ba}|a_n\rangle. \tag{$\widetilde{10.8}'$}$$

We note that (10.8), (10.8)′ in the diagonal case are consistent with

$$x^n_{aa} = |a_n\rangle o^n_{aa}\langle a_n|, \tag{$\widetilde{6.4}$}$$

$$o^n_{aa} = \langle a_n|x^n_{aa}|a_n\rangle, \tag{6.4}$$

as previously given under the equation numbers shown.

By use of (6.9) we obtain from (10.8)′ an expression for non-diagonal Young operators o^n_{ab} as the product of three diagonal operators separated by two permutations:

$$\begin{aligned} o^n_{ab} &= \langle a_n|a_n\rangle\langle a_n|(a_n|P|b_n)|b_n\rangle\langle b_n|b_n\rangle \\ &= K_{a/n}\langle a_a|a_n\rangle\langle a_n|(a_n|P|b_n)|b_n\rangle\langle b_n|b_b\rangle K_{b/n} \\ &= K_{a/n}o^a_{aa}x^n_{aa}P_{(a,a+1,\dots,n)}P_{(n,n-1,\dots,b)}x^n_{bb}o^b_{bb}K_{b/n} \\ &= K_{ab/nn}o^a_{aa}P_{(a,a+1,\dots,n)}o^n_{nn}P_{(n,n-1,\dots,b)}o^b_{bb}, \end{aligned} \tag{10.9}$$

where

$$K_{ab/nn} = K_{a/n}K_{b/n} = [ab/\{(a-1(b-1)\}]^{1/2}\{(n-1)/n\}. \tag{10.10}$$

By star transformation, the corresponding expressions for the non-diagonal Young operators $o^n_{a^*b^*}$ of $[2\ 1^{n-2}]$ are

$$o^n_{a^*b^*} = K_{ab/nn}o^a_{a^*a^*}P_{(a,a+1,\dots,n)}o^n_{n^*n^*}P_{(n,n-1,\dots,b)}o^b_{b^*b^*} \qquad (10.9)^*$$

$$= \langle a^*_n|(a_n|P|b_n)|b^*_n\rangle, \qquad (10.1)^*$$

since

$$(a^*_n|P|b^*_n) = (a_n|P|b_n), \tag{10.11}$$

by definition of tableau permutations and dual tableaux. The star transform of (10.7) is

$$x^n_{a^*b^*} = P_{(a,a+1,\dots,n)}o^n_{n^*n^*}P_{(n,n-1,\dots,b)}. \qquad (10.7)^*$$

Since the permutation P acting on the right or left of the standard Young operators form normalized linear combinations of Young operators according to the orthogonal matrix representation $[n-1, 1]$ or $[2\ 1^{n-2}]$, it follows that the ket–bra operators x^n_{ab}, $x^n_{a^*b^*}$ (diagonal and non-diagonal) are normalized linear combinations of standard bra–ket Young operators. Thus, for example, for $n = 3$, the four ket–bra operators for the representation [2 1] are given by

$$x^3_{33} = o^3_{33}, \tag{10.12}$$

$$x^3_{23} = P_{23}o^3_{33} = -(1/2)o^3_{33} + \sqrt{3}/2o^3_{23}, \tag{10.13}$$

$$x^3_{32} = o^3_{33}P_{23} = -(1/2)o^3_{33} + \sqrt{3}/2o^3_{32}, \tag{10.14}$$

$$\begin{aligned} x^3_{22} &= P_{23}o^3_{33}P_{23} = (P_{23}o^3_{33})(o^3_{33}P_{23}) = x^3_{23}x^3_{32} \\ &= (1/4)(o^3_{33} - \sqrt{3}o^3_{32} - \sqrt{3}o^3_{23} + 3o^3_{22}). \end{aligned} \tag{10.15}$$

It is to be noted that the linear combinations, although normalized, are not orthogonal.

We note finally that the expression (10.1) for o^n_{ab} simplifies in the two special limiting cases $a = n$, $b = n$. Thus, using

$$|n_n\rangle = \langle n_n|n_n\rangle = \langle n_n| \qquad (6.2), (\widetilde{6.2})$$

and

$$(a_n|P|n_n) = P_{(a,a+1,\dots,n)}, \tag{10.16}$$

we have

$$\begin{aligned} o^n_{an} &= \langle a_n|(a_n|P|n_n)|n_n\rangle \\ &= \langle a_n|P_{(a,a+1,\ldots,n)}\langle n_n|n_n\rangle = \langle a_n|a_n\rangle\langle a_n|P_{(a,a+1,\ldots,n)} \\ &= \langle a_n|P_{(a,a+1,\ldots,n)}, \end{aligned} \tag{10.17}$$

where use has been made of (10.3) and (6.3). So also, using

$$(n_n|P|a_n) = P_{(n,n-1,\ldots,a)}, \tag{10.18}$$

we have

$$\begin{aligned} o^n_{na} &= \langle n_n|(n_n|P|a_n)|a_n\rangle \\ &= \langle n_n|n_n\rangle P_{(n,n-1,\ldots,a)}|a_n\rangle = P_{(n,n-1,\ldots,a)}|a_n\rangle\langle a_n|a_n\rangle \\ &= P_{(n,n-1,\ldots,a)}|a_n\rangle. \end{aligned} \tag{10.19}$$

11. Proof that the Young operators multiply correctly

Since we have shown, in (8.7), that

$$|b_n\rangle\langle c_n| = \delta_{bc}|b_n\rangle\langle b_n| \tag{11.1}$$

it will be sufficient, with

$$o^n_{ab} = \langle a_n|(a_n|P|b_n)|b_n\rangle, \tag{11.2}$$

to establish the multiplication rule

$$o^n_{ab}o^n_{cd} = \delta_{bc}o^n_{ad}, \tag{11.3}$$

if we show that

$$o^n_{ab}o^n_{bc} = o^n_{ac}. \tag{11.4}$$

$$\text{LHS} = \langle a_n|(a_n|P|b_n)|b_n\rangle\langle b_n|(b_n|P|c_n)|c_n\rangle, \tag{11.5}$$

$$\text{RHS} = \langle a_n|(a_n|P|c_n)|c_n\rangle. \tag{11.6}$$

where LHS and RHS stand for left- and right-hand sides, respectively. We have shown, in (6.1)′, that

$$|b_n\rangle\langle b_n| = 2K_{/nn}S_{1\ldots\dot{b}\ldots n}A_{1b}S_{1\ldots\dot{b}\ldots n}. \tag{11.7}$$

Hence with

$$\langle a_n| = 2K_{/an}S_{1\ldots\dot{a}}A_{1a}S_{1\ldots\dot{a}\ldots n}, \tag{11.8}$$

$$|c_n\rangle = 2K_{/cn}S_{1\ldots\dot{c}\ldots n}A_{1c}S_{1\ldots\dot{c}}, \tag{11.9}$$

we have

$$\begin{aligned} \text{LHS} = 8K_{/ac}(K_{/nn})^2 & S_{1\ldots\dot{a}}A_{1a}S_{1\ldots\dot{a}\ldots n}(a_n|P|b_n)S_{1\ldots\dot{b}\ldots n}A_{1b} \\ & \times S_{1\ldots\dot{b}\ldots n}(b_n|P|c_n)S_{1\ldots\dot{c}\ldots n}A_{1c}S_{1\ldots\dot{c}} \\ &= 8K_{/ac}(K_{/nn})^2S_{1\ldots\dot{a}}(a_n|P|b_n)\underline{A_{1b}S_{1\ldots\dot{b}\ldots n}A_{1b}S_{1\ldots\dot{b}\ldots n}}A_{1b}(b_n|P|c_n)S_{1\ldots\dot{c}} \\ &= 4K_{/ac}K_{/nn}S_{1\ldots\dot{a}}(a_n|P|b_n)A_{1b}S_{1\ldots\dot{b}\ldots n}A_{1b}(b_n|P|c_n)S_{1\ldots\dot{c}}. \end{aligned} \tag{11.10}$$

$$\begin{aligned}\text{RHS} &= 4K_{/an}K_{/cn}S_{1\ldots\dot{a}}A_{1a}S_{1\ldots\dot{a}\ldots n}(a_n|P|c_n)S_{1\ldots\dot{c}\ldots n}A_{1c}S_{1\ldots\dot{c}} \\ &= 4K_{/ac}K_{/nn}S_{1\ldots\dot{a}}A_{1a}S_{1\ldots\dot{a}\ldots n}(a_n|P|b_n)(b_n|P|c_n)S_{1\ldots\dot{c}\ldots n}A_{1c}S_{1\ldots\dot{c}} \\ &= 4K_{/ac}K_{/nn}S_{1\ldots\dot{a}}(a_n|P|b_n)A_{1b}S_{1\ldots\dot{b}\ldots n}A_{1b}(b_n|P|c_n)S_{1\ldots\dot{c}} \\ &= \text{LHS},\end{aligned} \tag{11.11}$$

establishing (11.4) for $[n-1, 1]$. By star transformation, we have also, for $[2\ 1^{n-2}]$,

$$o^n_{a^*b^*}o^n_{c^*d^*} = \delta_{bc}o^n_{a^*d^*}. \tag{11.3)*$$

12. Verification of the orthogonal matrix elements for a transposition

In view of (11.4) it will be sufficient in establishing the standard orthogonal form (2.2), (2.3) of the representation $[n-1, 1]$ to show that

$$P_{a,a+1}o^n_{an} = (1/a)o^n_{an} + \{(a^2-1)^{1/2}/a\}o^n_{a+1,n}, \tag{12.1}$$

$$o^n_{na}P_{a,a+1} = (1/a)o^n_{na} + \{(a^2-1)^{1/2}/a\}o^n_{n,a+1}. \tag{12.2}$$

We have, from (10.17),

$$o^n_{an} = \langle a_n|P_{(a,a+1,a+2,\ldots,n)} = \langle a_n|P_{a,a+1}P_{(a+1,a+2,\ldots,n)}, \tag{12.3}$$

$$o^n_{a+1,n} = \langle (a+1)_n|P_{(a+1,a+2,\ldots,n)}. \tag{12.4}$$

Hence, cancelling the common term $P_{(a+1,a+2,\ldots,n)}$ on the right, (12.1) requires

$$P_{a,a+1}\langle a_n|P_{a,a+1} = (1/a)\langle a_n|P_{a,a+1} + \{(a^2-1)^{1/2}/a\}\langle (a+1)_n|. \tag{12.5}$$

With, from (5.1),

$$\langle a_n| = 2K_{/an}S_{1\ldots\dot{a}}A_{1a}S_{1\ldots\dot{a}\ldots n} \tag{12.6}$$

we deduce, by some cancellation,

$$\{(a^2-1)^{1/2}/a\}\langle (a+1)_n| = 2K_{/an}S_{1\ldots\dot{(a+1)}}A_{1(a+1)}S_{1\ldots\dot{(a+1)}\ldots n} \tag{12.7}$$

and also, noting that $S_{1\ldots\dot{a}} = S_{1\ldots(a-1)}$ is independent of a and $a+1$

$$P_{a,a+1}\langle a_n|P_{a,a+1} = 2K_{/an}S_{1\ldots\dot{a}}A_{1(a+1)}S_{1\ldots\dot{(a+1)}\ldots n}. \tag{12.8}$$

Hence, removing the common factor $2K_{/an}$, (12.5) requires

$$\begin{aligned}&S_{1\ldots\dot{a}}A_{1(a+1)}S_{1\ldots\dot{(a+1)}\ldots n} \\ &\quad= (1/a)S_{1\ldots\dot{a}}A_{1a}S_{1\ldots\dot{a}\ldots n}P_{a,a+1} + S_{1\ldots\dot{(a+1)}}A_{1(a+1)}S_{1\ldots\dot{(a+1)}\ldots n}.\end{aligned} \tag{12.9}$$

Now from the first part of (7.3), with $n=a$, we have

$$S_{1\ldots\dot{a}} - S_{1\ldots\dot{(a+1)}} = 2\{(a-1)/a\}S_{1\ldots\dot{a}}A_{1a}S_{1\ldots\dot{a}}. \tag{12.10}$$

Hence it remains to show (multiplying through by a)

$$2(a-1)S_{1\ldots\dot{a}}A_{1a}S_{1\ldots\dot{a}}A_{1(a+1)}S_{1\ldots\dot{(a+1)}\ldots n} = S_{1\ldots\dot{a}}A_{1a}S_{1\ldots\dot{a}\ldots n}P_{a,a+1}. \tag{12.11}$$

Cancelling the common factor $S_{1\ldots\dot{a}}$ on the left, putting

$$2A_{1a} = I - P_{1a}, \qquad (a-1)S_{1\ldots\dot{a}} = (I + P_{12} + P_{13} + \ldots + P_{1,a-1})S_{2\ldots\dot{a}} \tag{12.12}$$

and writing

$$A_{1a}S_{1\ldots\dot{a}\ldots n}P_{a,a+1}=P_{a,a+1}A_{1,a+1}S_{1\ldots(a\dot{+}1)\ldots n}, \quad (12.13)$$

it remains to show

$$(I-P_{1a})(I+P_{12}+P_{13}+\ldots+P_{1,a-1})S_{2\ldots\dot{a}}A_{1(a+1)}S_{1\ldots(a\dot{+}1)\ldots n} = P_{a,a+1}A_{1,a+1}S_{1\ldots(a\dot{+}1)\ldots n}. \quad (12.14)$$

Now, by (3.1),

$$S_{2\ldots\dot{a}}A_{1(a+1)}S_{1\ldots(a\dot{+}1)\ldots n}=A_{1(a+1)}S_{2\ldots\dot{a}}S_{1\ldots(a\dot{+}1)\ldots n}=A_{1(a+1)}S_{1\ldots(a\dot{+}1)\ldots n}, \quad (12.15)$$

so that we require

$$(I-P_{1a})(I+P_{12}+P_{13}+\ldots+P_{1,a-1})A_{1(a+1)}S_{1\ldots(a\dot{+}1)\ldots n} = P_{a,a+1}A_{1(a+1)}S_{1\ldots(a\dot{+}1)\ldots n}. \quad (12.16)$$

Now, for $r=2,3,\ldots,a-1$, we have

$$(I-P_{1a})P_{1r}A_{1(a+1)}S_{1\ldots(a\dot{+}1)\ldots n} = (I-P_{1a})P_{1r}A_{1(a+1)}P_{1r}S_{1\ldots(a\dot{+}1)\ldots n}=(I-P_{1a})A_{r(a+1)}S_{1\ldots(a\dot{+}1)\ldots n}. \quad (12.17)$$

Then since (for $r=2,3,\ldots,a-1$)

$$P_{1a}A_{r(a+1)}S_{1\ldots(a\dot{+}1)\ldots n}=A_{r(a+1)}P_{1a}S_{1\ldots(a\dot{+}1)\ldots n}=A_{r(a+1)}S_{1\ldots(a\dot{+}1)\ldots n}, \quad (12.18)$$

it follows

$$(I-P_{1a})(P_{12}+P_{13}+\ldots+P_{1,a-1})A_{1(a+1)}S_{1\ldots(a\dot{+}1)\ldots n}=0. \quad (12.19)$$

It remains to show, from (12.16), that

$$(I-P_{1a}-P_{a,a+1})A_{1(a+1)}S_{1\ldots(a\dot{+}1)\ldots n}=0, \quad (12.20)$$

or

$$A_{1,a,a+1}S_{1\ldots(a\dot{+}1)\ldots n}=0, \quad (12.21)$$

i.e.

$$A_{1,a,a+1}\underline{A_{1a}S_{1a}}S_{1\ldots(a\dot{+}1)\ldots n}=0, \quad (12.22)$$

which is true and hence (12.1) is established. (12.2) is established in a similar manner.

The starred relations

$$P_{a,a+1}o^n_{a^*n^*}=-(1/a)o^n_{a^*n^*}+\{(a^2-1)^{1/2}/a\}o^n_{(a+1)^*n^*} \quad (12.1)^*$$

$$o^n_{n^*a^*}P_{a,a+1}=-(1/a)o^n_{n^*a^*}+\{(a^2-1)^{1/2}/a\}o^n_{n^*(a+1)^*}, \quad (12.2)^*$$

are established in a similar manner (with S and A interchanged throughout) with the minus sign in front of $(1/a)$ being required because the final relation is

$$(I+P_{1a})S_{1(a+1)}A_{1\ldots(a\dot{+}1)\ldots n}=-P_{a,a+1}S_{1(a+1)}A_{1\ldots(a\dot{+}1)\ldots n}, \quad (12.23)$$

reducing to

$$S_{1,a(a+1)}\underline{S_{1a}A_{1a}}A_{1\ldots(a\dot{+}1)\ldots n}=0, \quad (12.24)$$

which is true as before. Multiplication by $o^n_{n^*b^*}$ on the right (for (12.1)*) or by $o^n_{a^*n^*}$ on the left (for (12.2)* with a replaced by b) leads to the standard relations (2.2)*, (2.3)*

Table 1. Young operators $\left(\begin{matrix}1\ldots\dot a\ldots n\\ a\end{matrix}\middle|0\middle|\begin{matrix}1\ldots\dot b\ldots n\\ b\end{matrix}\right)$ for $[n-1,1]$.

$a<n$	$2\sqrt{\left\{\frac{(a-1)(n-1)}{an}\right\}}S_{1\ldots\dot a}A_{1a}S_{1\ldots\dot a\ldots n}\left(\begin{matrix}1\ldots\dot a\ldots n\\ a\end{matrix}\middle\| P \middle\|\begin{matrix}1\ldots\dot b\ldots n\\ b\end{matrix}\right)A_{1b}S_{1\ldots\dot b}2\sqrt{\left\{\frac{(b-1)(n-1)}{bn}\right\}}$	$b<n$
	$2\sqrt{\left\{\frac{(a-1)(n-1)}{an}\right\}}S_{1\ldots\dot a}A_{1a}S_{1\ldots\dot a\ldots n}\left(\begin{matrix}1\ldots\dot a\ldots n\\ a\end{matrix}\middle\| P \middle\|\begin{matrix}1\ldots\dot n\\ n\end{matrix}\right)$	$b=n$
$a=n$	$\sqrt{\left\{\frac{2(n-1)}{n}\right\}}S_{1\ldots\dot n}A_{1n}\left(\begin{matrix}1\ldots\dot n\\ n\end{matrix}\middle\| P \middle\|\begin{matrix}1\ldots\dot b\ldots n\\ b\end{matrix}\right)S_{1\ldots\dot b}\sqrt{\left\{\frac{2(b-1)}{b}\right\}}$	$b<n$
	$\sqrt{\left\{\frac{2(n-1)}{n}\right\}}S_{1\ldots\dot n}A_{1n}\left(\begin{matrix}1\ldots\dot n\\ n\end{matrix}\middle\| P \middle\|\begin{matrix}1\ldots\dot n\\ n\end{matrix}\right)S_{1\ldots\dot n}\sqrt{\left\{\frac{2(n-1)}{n}\right\}}$	$b=n$

Examples

$$o^n_{ab}=\left(\begin{matrix}1\ldots\dot a\ldots n\\ a\end{matrix}\middle|o\middle|\begin{matrix}1\ldots\dot b\ldots n\\ n\end{matrix}\right)=4\left(\frac{n-1}{n}\right)\sqrt{\left\{\frac{(a-1)(b-1)}{ab}\right\}}S_{1\ldots\dot a}A_{1a}S_{1\ldots\dot a\ldots n}\left(\begin{matrix}1\ldots\dot a\ldots n\\ a\end{matrix}\middle|P\middle|\begin{matrix}1\ldots\dot b\ldots n\\ b\end{matrix}\right)A_{1b}S_{1\ldots\dot b}$$

$$o^n_{an}=\left(\begin{matrix}1\ldots\dot a\ldots n\\ a\end{matrix}\middle|o\middle|\begin{matrix}1\ldots\dot n\\ n\end{matrix}\right)=2\sqrt{\left\{\frac{(a-1)(n-1)}{an}\right\}}S_{1\ldots\dot a}A_{1a}S_{1\ldots\dot a\ldots n}\left(\begin{matrix}1\ldots\dot a\ldots n\\ a\end{matrix}\middle|P\middle|\begin{matrix}1\ldots\dot n\\ n\end{matrix}\right)$$

$$o^n_{nb}=\left(\begin{matrix}1\ldots\dot n\\ n\end{matrix}\middle|o\middle|\begin{matrix}1\ldots\dot b\ldots n\\ b\end{matrix}\right)=2\sqrt{\left\{\frac{(b-1)(n-1)}{bn}\right\}}S_{1\ldots\dot n}A_{1n}\left(\begin{matrix}1\ldots\dot n\\ n\end{matrix}\middle|P\middle|\begin{matrix}1\ldots\dot b\ldots n\\ b\end{matrix}\right)S_{1\ldots\dot b}=2\sqrt{\left\{\frac{(b-1)(n-1)}{bn}\right\}}\left(\begin{matrix}1\ldots\dot n\\ n\end{matrix}\middle|P\middle|\begin{matrix}1\ldots\dot b\ldots n\\ b\end{matrix}\right)S_{1\ldots\dot b\ldots n}A_{1b}S_{1\ldots\dot b}$$

$$o^n_{nn}=\left(\begin{matrix}1\ldots\dot n\\ n\end{matrix}\middle|o\middle|\begin{matrix}1\ldots\dot n\\ n\end{matrix}\right)=\frac{2(n-1)}{n}S_{1\ldots\dot n}A_{1n}S_{1\ldots\dot n}$$

Table 2. Young operators $\left(\begin{array}{c|c|c} 1a & & 1b \\ \vdots & & \vdots \\ \dot{a} & 0 & \dot{b} \\ \vdots & & \vdots \\ n & & n \end{array}\right)$ for $[2\,1^{n-2}]$.

$a<n$	$2\sqrt{\left\{\frac{(a-1)(n-1)}{an}\right\}}A_{1\ldots\dot{a}}S_{1a}A_{1\ldots\dot{a}\ldots n}\left(\begin{array}{c\|}1a\\ \vdots\\ \dot{a}\\ \vdots\\ n\end{array}\right. P$	$\left\|\begin{array}{c}1b\\ \vdots\\ \dot{b}\\ \vdots\\ n\end{array}\right) S_{1b}A_{1\ldots\dot{b}}2\sqrt{\left\{\frac{(b-1)(n-1)}{bn}\right\}}$	$b<n$
		$\left\|\begin{array}{c}1n\\ \vdots\\ \dot{n}\end{array}\right)$	$b=n$
$a=n$	$\sqrt{\left\{\frac{2(n-1)}{n}\right\}}A_{1\ldots\dot{n}}S_{1n}\left(\begin{array}{c\|}1n\\ \vdots\\ \dot{n}\end{array}\right. P$	$\left\|\begin{array}{c}1b\\ \vdots\\ \dot{b}\\ \vdots\\ n\end{array}\right) A_{1\ldots\dot{b}}\sqrt{\left\{\frac{2(b-1)}{b}\right\}}$	$b<n$
		$\left\|\begin{array}{c}1n\\ \vdots\\ \dot{n}\end{array}\right) A_{1\ldots\dot{n}}\sqrt{\left\{\frac{2(n-1)}{n}\right\}}$	$b=n$

Examples

$$o^n_{a*b*}=(1a*\ldots\dot{a}\ldots n|o|1b*\ldots\dot{b}\ldots n)=4\left(\frac{n-1}{n}\right)\sqrt{\left\{\frac{(a-1)(b-1)}{ab}\right\}}A_{1\ldots\dot{a}}S_{1a}$$

$$\times A_{1\ldots\dot{a}\ldots n}\left(\begin{array}{c}1\ldots\dot{a}\ldots n\\ a\end{array}\middle|P\middle|\begin{array}{c}1\ldots\dot{b}\ldots n\\ b\end{array}\right)S_{1b}A_{1\ldots\dot{b}}$$

$$o^n_{a*n*}=(1a*\ldots\dot{a}\ldots n|o|1n*\ldots\dot{n})=2\sqrt{\left\{\frac{(a-1)(n-1)}{an}\right\}}A_{1\ldots\dot{a}}S_{1a}A_{1\ldots\dot{a}\ldots n}\left(\begin{array}{c}1\ldots\dot{a}\ldots n\\ a\end{array}\middle|P\middle|\begin{array}{c}1\ldots\dot{n}\\ n\end{array}\right)$$

$$o^n_{n*b*}=(1n*\ldots\dot{n}|o|1b*\ldots\dot{b}\ldots n)=2\sqrt{\left\{\frac{(b-1)(n-1)}{bn}\right\}}A_{1\ldots\dot{n}}S_{1n}\left(\begin{array}{c}1\ldots\dot{n}\\ n\end{array}\middle|P\middle|\begin{array}{c}1\ldots\dot{b}\ldots n\\ b\end{array}\right)A_{1\ldots\dot{b}}$$

$$=2\sqrt{\left\{\frac{(b-1)(n-1)}{bn}\right\}}\left(\begin{array}{c}1\ldots\dot{n}\\ n\end{array}\middle|P\middle|\begin{array}{c}1\ldots\dot{b}\ldots n\\ b\end{array}\right)A_{1\ldots\dot{b}\ldots n}S_{1b}A_{1\ldots\dot{b}}$$

$$o^n_{n*n*}=(1n*\ldots\dot{n}|o|1n*\ldots\dot{n})=\frac{2(n-1)}{n}A_{1\ldots\dot{n}}S_{1n}A_{1\ldots\dot{n}}$$

for $[2\ 1^{n-2}]$. We note here that since the star transformation was defined to multiply each permutation by its parity and hence the transposition $P_{a,a+1}$ by -1, we could have expected here a minus sign also before the second terms on the right, but Young's standard convention, to which we adhere, makes these terms positive in all cases. (This is an arbitrary choice of phase which does not affect the orthogonality of the representation.)

13. Results in tabular form

The results are presented in tables 1 and 2.

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